Sign test

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Statistics for Business

Sign test

Definition

Sign test is a non-parametric test, a special case for the binomial test with $p = 1/2$, with these applications:

- for a simple sample, testing is the population has a given value for the median; e.g., given the sample $3.1, 2.4, 5.6, 6.7, 4.2, 3.8$, we may test by the sign test if the sample is compatible with a population with $Me = 3.5$.
- for paired samples, testing is the difference of medians for both populations us 0, that is, if one population takes bigger values than the other one. E.g., after applying a given fertilizer, to test is a plant gives better returns.
- Setting confidence intervals for the median.

- Non-parametric tests are tests that don't set a given distribution for the population, so they are more flexible.
- By means of a binomial test, we decide if a given p probability is compatible with data.

Sign test

Test about the median (I)

- The median is the value that leaves below it 50% of data, that is, a 0.5 probability.
- Hence, in a population the probability for one data being below the median is 0.5.
- To perform the test, variable must be continuous, that is, it must take many different values.
- Under the null hypothesis, we set a given value for the median: $H_0 : Me = m$. Alternatively, $H_a : Me \neq m$.

Test about the median (II)

- We assign the $-$ sign to the data below $Me = m$; to the data above $Me = m$, we assign the $+$ sign.
- We count $-$ and $+$ data: r^- and r^+.
- This test is two-sided: we reject the null hypothesis, $H_0 : Me = m$, when r^- and r^+ values are very big and very small. In order to accept H_0, r^- and r^+ should be close to the number of data divided by 2.
- But, in order to perform the test in a standard way, we take always the smallest value between r^- and r^+. We name this value r.
- As is the smallest value, we reject $H_0 : Me = m$ when r is small enough.

Test about the median (III)

- For small samples, we look for the r^* critical value into the tables, for different values of the α significance level.
- We reject the null hypothesis, that is, the value given for the median, when we have $r \leq r^*$.
- As the variable is continuous, in theory there is no data equal to the median value, but if it were, we would remove it.

Test about the median (IV): calculating p value

- The probability for one data of being below of above the median in the null hypothesis is 0.5.
- For n data, the number of data below or above the median is distributed $B(n, 0.5)$.
- For n data, the smallest number of signs will be x when data below median are x or data above median are x.
- And the probability for that event is:

$$\begin{aligned} P[min = x] &= P[above\ Me = x] + P[below\ Me = x] \\ &= 0.5^x 0.5^{n-x} \frac{n!}{x!(n-x)!} + 0.5^x 0.5^{n-x} \frac{n!}{x!(n-x)!} \\ &= 2 \times 0.5^n \frac{n!}{x!(n-x)!} \end{aligned}$$

- The p-value is the probability for the evidence or something stranger. As we took the minimum value, "the strange thing" is on the lower side. So:

$$p = P[X \leq x] = 2 \sum_{i=1}^{x} 0.5^n \frac{n!}{i!(n-i)!} = 2 \times 0.5^n \sum_{i=1}^{x} \frac{n!}{i!(n-i)!}$$

Test about the median (V): example

- Data: 6.14-6.45-8.33-11.05-5.67-5.22-7.52-10.08-12.34
- We want to test if the median in the population is 7: $H_0 : Me = 7$.
  We set $\alpha = 10\%$.
- We set the signs: $-, -, +, +, -, -, +, +, +$.
- $r^- = 4; r^+ = 5$. We take the smallest one: $r = 4$.
- We look for the critical value into the table: $r^* = 1$.
- As r is above that value, we accept the null hypothesis, that is, that median is 7.
- The p-value is the probability of having 4 data below the median (and not above the median):

$$p = P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4]$$
$$= 0.5^0 \cdot 0.5^9 \cdot \frac{9!}{0!9!} + 0.5^1 \cdot 0.5^8 \cdot \frac{9!}{1!8!} + 0.5^2 \cdot 0.5^7 \cdot \frac{9!}{2!7!}$$
$$+ 0.5^3 \cdot 0.5^6 \cdot \frac{9!}{3!6!} + 0.5^4 \cdot 0.5^5 \cdot \frac{9!}{4!3!} = \sum_{i=1}^{4} 0.5^9 \frac{9!}{i!(9-i)!} = 0.5$$

- Here we compare the p-value to $\alpha/2$, as we have not taken into account that the smallest r may also be the number of signs above the median. As we have $p > \alpha/2$, we accept $H_0 : Me = 7$.

Test about the median (VI): one-sided test

- Data: 1.34-2.45-0.33-4.73-0.68-1.02-3.13-3.45
- We want to test if median is bigger than 0.5: $H_0 : Me \geq 0.5$. $\alpha = 10\%$
- We take the median as 0.5 (boundary value),and assign signs to data: $+,+,-,+,+,+,+,+ \rightarrow r^- = 1; r^+ = 7$
- We reject $H_0 : Me > 0.5$ when r^- is very big (r^+ very small).
- We look at the smallest, in order to use the tables. So in this case we have to look at the r^+ value.
- We take the critical value from the table (we have to multiply α by 2)*: $n = 8, \alpha = 0.20 \rightarrow r^* = 1$
- The value for the statistic, $r^+ = 7$, is bigger than the critical value, $r^* = 1$, so we accept that median is bigger than 0.5.

* One-sided tests with α significance levels and two-sided tests with 2α significance levels have the same critical values. As tables are given for two-sided tests, we multiply α by 2. See the annex.

Test about the median (VII): one-sided test

A summary:

- for $H_0 : Me < m$, look at r^-.
- for $H_0 : Me > m$, look at r^+.
- Then, look for the r^* critical value on tables, after multiplying α by 2, and take the decision.

Paired samples (I)

- We say that two samples are paired when they are taken over the same elements, before and after a treatment.
- E.g., we have paired samples when we measure the quantity of a substance into the blood before and after taking a drug for the same group of patients; and when we take califications over the same students at the beginning and end of the course.
- Under H_0 the median difference is 0 for both populations, before and after the treatment (roughly speaking, both populations are the same). Alternatively, we decide if that median is bigger or smaller than 0.
- Variable must be continuous, in order to avoid ties.

Paired samples (II)

- Calculate differences for both samples values.
- If the difference is positive, write $+$; if negative, write $-$.
- Are those values compatible with the alternative hypothesis? If not, accept the null hypothesis, and the test is finished.
- Otherwise, in order to know if we may reject the null hypothesis and accept the alternative one, take the smallest value between r^+ and r^-, name it r , and compare it to the r^* critical value into the tables.
- If $r < r^*$, we reject both populations are the same, and accept the effect claimed in the alternative hypothesis. Otherwise, there is no significant difference for medians before and after the treatment.

Paired samples (III): example

- In a class we have measured mathematical ability at the beginning and the end of the course:
- Beginning: 4.2-6.5-7.2-5.4-8.4-5.6-7.4-7.2
- End: 5.4-6.6-7.4-7.0-8.2-6.8-7.3-7.8
- Question: have the students bigger ability after the course? $\alpha = 0.10$
- Assign signs after calculating differences:++++-+-+
- 6 positive and 2 negative. So, we may claim that course increased the ability. Otherwise, we couldn't claim it and the test would be over.
- We take the smallest: $r = 2$.
- We take the critical values into the tables:

$$n = 8, \alpha = 0.20(2 - sided) \rightarrow r^* = 1$$

- We have $r > r^*$, so we accept that the median difference is 0, and cannot claim that the course was profitable (we should have 1 negative at most for that).

Applying the sign test: confidence interval about the median

Steps with example

- Give 95% confidence interval about the median for these data:
- Ordered data: 6.14-6.45-8.33-11.05-5.67-5.22-7.52-10.08-12.34
- Sample size: 9.
- Give cumulated probabilities for $B(n = 9, p = 0.5)$ distribution.
- $(x = 0, p = 0.0009)$;$(x = 1, p = 0.01)$;$(x = 2, p = 0.054)$;...
- Take the value that leaves *at most* a 2.5% probability (non confidence level, 5%, divided by 2, because the interval is symmetric). At most, because we take the confidence level at least.
- That value is $x = 1$. We take $x + 1 = 2$.
- We take the $x + 1 =$ 2nd data fron the beginning and the end.
- So, with 98% confidence level $(1 - 0.01 \times 2)$, we can claim that median is between 6.45 and 10.08.

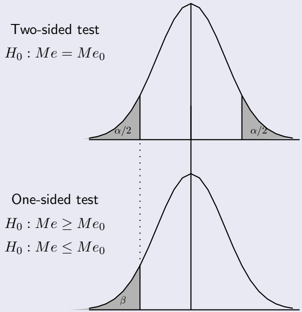
Sign test

Big samples

When n sample size is big, we are out of the table when we look for critical values. In such cases, we have to apply the De Moivre-Laplace theorem (in a following lesson - Normal distribution and stochastic convergence), using the normal distribution as an approximation for the binomial distribution.

Remark: we will probably solve a problem as an example for such cases in that lesson.

Annex 1: Why we take 2α in tables for one-sided sign tests



$$\beta = \alpha/2 \rightarrow \alpha = 2\beta$$